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A NEW TOPOLOGY FROM AN OLD ONE

Halgwrd Mohammed Darwesh*

Abstract. In the present paper we construct and introduce a new topology from an old one which are independent each of the other. The members of this topology are called ω_δ-open sets. We investigate some basic properties and their relationships with some other types of sets. Furthermore, a new characterization of regular and semi-regular spaces are obtained. Also, we introduce and study some new types of continuity, and we obtain decompositions of some types of continuity.

1. Introduction

A subset A of a topological space X is called regular open [18] if $A = intClA$. The collection of all regular open subsets of a topological space (X, τ) forms a base for a topology τ_s on X coarser than τ, (X, τ_s) is called the semiregularization of (X, τ). In 1968, Veličko [20], has defined δ-open and θ-open sets to investigate some characterizations of H-closed spaces, and he showed that the collection of all θ-open and δ-open subsets of a topological space (X, τ) form topologies on X which are denoted by τ_θ and τ_δ, respectively. It is well known that $\tau_s = \tau_\delta$ and $\tau_\theta \subseteq \tau_\delta \subseteq \tau$. In 1982, Hdeib [7] introduced the notations of ω-closedness and ω-openness. The collection of all ω-open subsets of a space (X, τ) is a topology on X which is denoted by τ^ω and it is finer than τ. Al-Hawary et. al. [1] and Ekici et. al. [5] have introduced the concepts of ω^o-open and ω_θ-open sets, respectively. Also, they showed that the collection of all ω^o-open sets $\omega^o O(X)$ and the collection of all ω_θ-open sets $\omega_\theta O(X)$ are topologies on X such that $\tau \subseteq \omega_\theta O(X) \subseteq \omega^o O(X) \subseteq \tau^\omega$. In Section 2, we will offer topology on X by utilizing the new notion of sets which we call ω_δ-open sets. This topology is strictly finer than each of $\omega_\theta O(X)$ and the

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semiregularization topology of X and it is strictly coarser than $\omega^o O(X)$. Moreover, it is independent of τ. Furthermore, we introduce some other new notions of sets and we will obtain new characterizations of regular and semi-regular spaces. After Levine's decomposition of continuity [9], authors in topological spaces have defined some type of continuity and they obtained some decompositions of some types of continuity such as [2, 19, 3, 6]. In section 3, we introduce a new type of continuity by using the concept of ω_δ-open sets and some other weaker forms of continuity and we obtain some decompositions of some types of continuity.

2. Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure and interior of A are denoted by ClA and $intA$, respectively.

Definition 2.1. A subset G of a topological space (X, τ) is called δ-open (resp. θ-open) set [20] if for each $x \in G$, there exists an open set O containing x such that $intClO \subseteq G$ (resp. $ClO \subseteq G$).

For a subset A of a space X, the $int_\delta A$ and $int_\theta A$ will be denoted the δ-interior and θ-interior of A, respectively.

Definition 2.2. A space (X, τ) is said to be semi-regular [15] if $\tau = \tau_\delta$.

Theorem 2.3. A space (X, τ) is regular [11] if and only if $\tau = \tau_\theta$.

Definition 2.4. A subset U of a topological space (X, τ) is called ω-open [7] (resp. ω^o-open [1] and ω_θ-open [5])set if for each $x \in U$, there exists an open set O containing x such that $O - U$ (resp. $O - intU$ and $O - int_\theta U$) is countable.

The ω-interior of a subset A of a space X is denoted by $int_\omega A$.

3. A new topology

Definition 3.1. A subset U of a topological space (X, τ) is called ω_δ-open if for each $x \in U$, there exists an open set G containing x such that $G - int_\delta U$ is countable. The complement of ω_δ-open sets are called

ω_δ-closed sets. The family of all ω_δ-open (resp. ω_δ-closed) subset of a space X are denoted by $\omega_\delta O(X)$ (resp. $\omega_\delta C(X)$).

It is easy to see that each clopen, θ-open, ω_θ-open, regular open and δ-open sets are ω_δ-open and each ω_δ-open set is ω^o-open and ω-open, but not conversely. The following examples support our claim.

EXAMPLE 3.2. Let $X=\{a,b\}$ and $\tau=\{\phi,\{a\},X\}$. Then $\tau_\delta=\{\phi,X\}=\tau_\theta$ and $\omega_\delta O(X)=\{\phi, \{a\}, \{b\}, X\}=\omega_\theta O(X)=\omega^o(X)=\tau^\omega$. Thus the set $A=\{b\}$ is ω_δ-open, ω_θ-open, ω^o-open and ω-open but it is neither clopen, θ-open, δ-open nor open.

EXAMPLE 3.3. Let the set of all real number $\mathbb{R}$ be equipped with the topology $\tau=\{\phi,[0,1], \mathbb{R}\}$. The set $A=[0,1]$ is open, ω^o-open and ω-open but it is neither ω_δ-open, θ-open nor δ-open.

EXAMPLE 3.4. Let the set of all real number $\mathbb{R}$ be equipped with the topology $\tau=\{\phi,\mathbb{Q}^c,\mathbb{Q}\cap(0,1),\mathbb{Q}^c\cup(\mathbb{Q}\cap(0,1)),\mathbb{R}\}$, where $\mathbb{Q}^c$ and $\mathbb{Q}$ are denoted the set of all irrational and rational numbers, respectively. Then the set $A=\mathbb{Q}^c$ is ω_δ-open set but not ω_θ-open.

From the above examples, we conclude that the concepts of ω_δ-open sets and open sets are independent topological concepts.

THEOREM 3.5. Let U be a subset of a space X. Then U is ω_δ-open if and only if for each $x\in U$, there exists an open set G containing x and a countable set C such that $G-C\subseteq int_\delta U$.

Proof. Let U be an ω_δ-open subset of X and let x be any element of U. Then by Definition 3.1, there exists an open set G containing x such that the set $C=G-int_\delta U$ is countable. Therefore, $G-C\subset int_\delta U$. Conversely; let $x\in U$. Then by hypothesis, there exists an open set G containing x and a countable set C such that $G-C\subseteq int_\delta U$. Thus $G-int_\delta U\subseteq C$, this means that $G-int_\delta U$ is countable. Hence U is an ω_δ-open set. □

THEOREM 3.6. For any space X, the family $\omega_\delta O(X)$ forms a topology on X.

Proof. Since ϕ and X are δ-open subsets of X and each δ-open set is ω_δ-open, then ϕ, $X\in\omega_\delta O(X)$. Let $U,V\in\omega_\delta O(X)$ and $x\in U\cap V$. Then there exist open sets G and O both containing x such that $G-int_\delta U$ and $O-int_\delta V$ are countable sets. Since $G\cap O-int_\delta(U\cap V)\subseteq(G-int_\delta U)\cup(O-int_\delta V)$, then $G\cap O-int_\delta(U\cap V)$ is a countable subset of X. Hence $U\cap V\in\omega_\delta O(X)$. Let $\{U_\lambda;\lambda\in\Lambda\}\subseteq\omega_\delta O(X)$

and $x \in \bigcup\{U_\lambda; \lambda \in \Lambda\ \}$. Then there exist $\lambda_0 \in \Lambda$ and an open set G containing x such that $x \in U_{\lambda_0}$ and $G - int_\delta U_{\lambda_0}$ is countable. Since $G - int_\delta(\bigcup\{U_\lambda; \lambda \in \Lambda\}) \subseteq G - int_\delta U_{\lambda_0}$, then $\bigcup\{U_\lambda; \lambda \in \Lambda\} \in \omega_\delta O(X)$. Thus $\omega_\delta O(X)$ is a topology on X. □

Remark 3.7. From what we have done above, we notice that for any space (X, τ), $\tau_\theta \subset \tau_\delta \subset \omega_\delta O(X) \subset \omega^o O(X) \subset \tau^\omega$ and $\tau_\theta \subseteq \omega_\theta O(X) \subset \omega_\delta O(X) \subset \omega^o O(X) \subset \tau_\omega$. Further, τ and $\omega_\delta O(X)$ are independent topologies on X.

Proposition 3.8. Let (X, τ) be a topological space. Then

(1) (X, τ) is locally countable if and only if $\omega_\delta O(X)$ is a discrete topology on X.

(2) (X, τ) is an anti-locally countable space if and only if $(X, \omega_\delta O(X))$ is anti-locally countable.

Proof. (1) Let A be any subset of X and $x \in A$. Since X is locally countable, then there exists a countable open subset G of X contains x. Since $G - int_\delta A \subseteq G$, then $G - int_\delta A$ is countable. Thus A is ω_δ-open. Hence in view of Theorem 3.6, $\omega_\delta O(X)$ is a discrete topology on X. The converse part is obvious.

(2) Let (X, τ) be an anti-locally countable space. To show $(X, \omega_\delta O(X))$ is anti-locally countable. On contrary, we suppose that $(X, \omega_\delta O(X))$ is not anti-locally countable. Then there exists a countable non-empty ω_δ-open subset U of X. Therefore, there exists a point $x \in U$ and by Definition 3.1, there exists an open set G containing x such that $G - int_\delta U$ is countable. But since $G - U \subseteq G - int_\delta U$, then $G - U$ is countable. Since $G = U \cup (G - U)$, then G is a non-empty countable open subset of X, this is impossible. Conversely, let (X, τ) be a space for which $(X, \omega_\delta O(X))$ is an anti-locally countable space. We suppose that (X, τ) is not anti-locally countable space. Then there exists a non-empty countable open subset G of X. Let x be any point of G. Since $G - int_\delta G \subseteq G$, then $G - int_\delta G$ is countable and hence G is a non-empty countable ω_δ-open subset of X which is a contradiction to our hypothesis. Hence (X, τ) is anti-locally countable. □

Theorem 3.9. If (X, τ) is a Lindelöf space, then so is $(X, \omega_\delta O(X))$.

Proof. Let $\Psi = \{V_\lambda; \lambda \in \Lambda\ \}$ be any ω_δ-open cover of X. Then for each $x \in X$, there exists $\lambda_x \in \Lambda$ and an open set G_{λ_x} containing x such that $C_{\lambda_x} = G_{\lambda_x} - int_\delta V_{\lambda_x}$ is countable. Let $\Lambda_X = \{\lambda_x \in \Lambda; x \in X\}$. Then $\{G_{\lambda_x}; \lambda_x \in \Lambda_X\ \}$ is an open cover of X. Since X is Lindelöf, then there exists a countable subset Λ_0 of Λ_X (hence of Λ)

such that $\{G_\lambda; \lambda \in \Lambda_0\}$ covers X. Therefore, the family $\{V_\lambda; \lambda \in \Lambda_0\}$ covers X except the countable subset $C = \cup\{C_\lambda; \lambda \in \Lambda_0\}$ of X. Since C is countable, then it is clear that there exists a countable subset Λ_1 of Λ such that $\{V_\lambda; \lambda \in \Lambda_1\}$ covers C, and hence $\{V_\lambda; \lambda \in \Lambda_1 \cup \Lambda_0\}$ is a countable subcover of Ψ. Thus $(X, \omega_\delta O(X))$ is Lindelöf. $\square$

The following example shows that the converse of the above theorem is not true in general.

Example 3.10. Consider the particular uncountable point topology $\Im = \{G \subseteq \mathbb{R}; (0,1) \subseteq G\} \cup \{\phi\}$ on $\mathbb{R}$ ([17], Example 10, p. 44). Since the open cover $\{(0,1) \cup \{p\}; p \in \mathbb{R} - (0,1)\}$ of $\mathbb{R}$ has no any countable subcover, so $(\mathbb{R}, \Im)$ is not Lindelöf. But Since $\Im_\delta$=$\{\phi, \mathbb{R}\}$, so it is easy to see that $\omega_\delta O(\mathbb{R})$=$\{\phi, \mathbb{R}\}$, and hence $(\mathbb{R}, \omega_\delta O(\mathbb{R}))$ is Lindelöf.

Recall that a space X is said to be a nearly Lindelöf space [4] if every regular open cover of X has a countable subcover.

Proposition 3.11. Let (X, τ) be a space such that $(X, \omega_\delta O(X))$ is Lindelöf. Then (X, τ) is nearly Lindelöf.

Proof. Obvious. $\square$

The following example shows that the converse of the above proposition is not true in general.

Example 3.12. Consider the finite particular point topology $\tau = \{G \subseteq \mathbb{R}; 0 \in G\} \cup \{\phi\}$ on $\mathbb{R}$ ([17], Example 8, p. 44). Since τ_δ=$\{\phi, \mathbb{R}\}$, then this space is nearly Lindelöf. But since the space $(\mathbb{R}, \tau)$ is a locally countable space, then by part (1) of Proposition 3.8, $(\mathbb{R}, \omega_\delta O(\mathbb{R}))$ is an uncountable discrete space, and hence it is not Lindelöf.

Recall that a space X is said to be nearly compact [16] if each regular open cover of X has a finite subcover. Then it is easy to see that for any space (X, τ), compactness of $(X, \omega_\delta O(X))$ implies nearly compactness of the space (X, τ). But not conversely as the following example shows:

Example 3.13. Let the set of all natural numbers $\mathbb{N}$ be equipped with the indiscrete topology $\Im_{ind}$. Then $(\mathbb{N}, \Im_{ind})$ is both compact and nearly compact space. But since $(\mathbb{N}, \Im_{ind})$ is locally-countable, then by part (1) of Proposition 3.8, $\omega_\delta O(\mathbb{N})$ is the discrete topology on $\mathbb{N}$ and hence $(\mathbb{N}, \omega_\delta O(\mathbb{N}))$ is not compact.

Example 3.14. Let the set of all real numbers $\mathbb{R}$ be equipped with the co-countable topology τ_{coc}. Then it is clear that $(\mathbb{R}, \tau_{coc})$ is not compact. But since $\omega_\delta O(\mathbb{R}) = \{\phi, X\} = \tau_\delta$, then $(\mathbb{R}, \omega_\delta O(\mathbb{R}))$ is compact.

The last two examples show that the compactness of a space (X, τ) is neither imply nor implied by the compactness of the space $(X, \omega_\delta O(X))$.

DEFINITION 3.15. Let A be a subset of a topological space X. Then

(1) The intersection of all ω_δ-closed subsets of X containing A is called ω_δ-closure of A and it is denoted by $\omega_\delta ClA$.

(2) The union of all ω_δ-open subsets of X contained in A is called ω_δ-interior of A and it is denoted by $\omega_\delta intA$.

LEMMA 3.16. Let A and B be any subsets of a topological space X. Then

(1) $\omega_\delta ClA \subseteq Cl_\delta A$ and $int_\delta A \subseteq \omega_\delta intA$.

(2) If $A \subseteq B$, then $\omega_\delta ClA \subseteq \omega_\delta ClB$ and $\omega_\delta intA \subseteq \omega_\delta intB$.

(3) $x \in \omega_\delta ClA$ if and only if $A \cap U \neq \phi$ for each ω_δ-open set U containing x, and $x \in \omega_\delta intA$ if and only if there exists an ω_δ-open set U such that $x \in U \subseteq A$.

(4) $\omega_\delta ClA \in \omega_\delta C(X)$ and $\omega_\delta intA \in \omega_\delta O(X)$.

(5) A is ω_δ-open (resp. ω_δ-closed) if and only if $A = \omega_\delta intA$ (resp. $A = \omega_\delta ClA$).

(6) $\omega_\delta Cl(\omega_\delta ClA) = \omega_\delta ClA$ and $\omega_\delta int(\omega_\delta intA) = \omega_\delta intA$.

(7) $\omega_\delta Cl(X - A) = X - \omega_\delta intA$ and $\omega_\delta int(X - A) = X - \omega_\delta ClA$.

(8) $\omega_\delta int(A \cap B) = \omega_\delta intA \cap \omega_\delta intB$ and $\omega_\delta Cl(A \cup B) = \omega_\delta ClA \cup \omega_\delta ClB$.

DEFINITION 3.17. A subset U of a topological space X is said to be ω_δ^δ-open (resp. ω_δ^o-open, ω_δ^θ-open and ω_δ^ω-open), if $\omega_\delta intU = int_\delta U$. (resp. $\omega_\delta intU = intU$, $\omega_\delta intU = int_\theta U$ and $\omega_\delta intU = int_\omega U$).

REMARK 3.18. It is easy to see that

(1) Every ω_δ-open set is ω_δ^ω-open.

(2) Every δ-open set is ω_δ^δ-open, ω_δ^o-open and ω_δ^ω-open.

(3) Every θ-open set is ω_δ^θ-open, ω_δ^δ-open, ω_δ^o-open and ω_δ^ω-open.

(4) Every ω_δ^θ-open set is ω_δ^δ-open.

In Example 3.4 the set $\mathbb{Q}^c$ is ω_δ^δ-open but not ω_δ^θ-open. This with the following examples show that the converse of neither parts of the above remark are true.

EXAMPLE 3.19. Consider the topological space $(\mathbb{R}, \tau)$, where $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^c\}$ and the set of all natural numbers $\mathbb{N}$. Since $\omega_\delta int\mathbb{N} = \phi = int_\delta\mathbb{N} = int_\theta\mathbb{N} = int\mathbb{N} = int_\omega\mathbb{N}$, then $\mathbb{N}$ is ω_δ^ω-open, ω_δ^δ-open, ω_δ^o-open and ω_δ^θ-open. But it is neither open, δ-open, θ-open, ω_δ-open nor ω-open. However, the set $\mathbb{Q}^c$ is ω-open but not ω_δ^ω-open.

REMARK 3.20. The first and second components of each of the following order pairs are independent, (open, ω_δ^δ-open), (open, ω_δ^o-open), (open, ω_δ^θ-open), (ω_δ^δ-open,ω_δ^o-open), (ω_δ^δ-open,ω_δ^ω-open), (ω_δ-open, ω_δ^δ-open), (ω_δ-open, ω_δ^o-open), (ω_δ-open, ω_δ^θ-open), (ω_δ^o-open, ω_δ^ω-open) and (ω-open, ω_δ^ω-open):

(1) In Example 3.2, the set $A = \{a\}$ is ω_δ-open, open, ω-open, ω_δ^o-open and ω_δ^ω-open but it is neither ω_δ^θ-open nor ω_δ^δ-open. However, the set $B = \{b\}$ is ω_δ-open but it is neither ω_δ^δ-open nor ω_δ^o-open.
(2) In Example 3.14, the set $A = \mathbb{R} - \mathbb{N}$ is ω_δ^δ-open but it is neither ω_δ^o-open nor ω_δ^ω-open.
(3) In the usual space $(\mathbb{R}, \tau)$, the set $A = \{0\}$ is ω_δ^δ-open, ω_δ^o-open, ω_δ^θ-open and ω_δ^ω-open but it is neither open, δ-open, θ-open, ω-open nor ω_δ-open.

Thus we obtain the following diagram

$$
\begin{array}{ccccccc}
\omega_\delta^o - open & & regular\ open & & open & & \omega - open \\
 & \nwarrow & \downarrow & \nearrow & & \searrow & \uparrow \\
\omega_\delta^\delta - open & \leftarrow & \delta - open & \rightarrow & \omega_\delta - open & \rightarrow & \omega^o - open \\
\uparrow & & \uparrow & & \uparrow & \searrow & \\
\omega_\delta^\theta - open & \leftarrow & \theta - open & \rightarrow & \omega_\theta - open & & \omega_\delta^\omega - open \\
 & & \uparrow & & & & \\
 & & clopen & & & &
\end{array}
$$

THEOREM 3.21. Let U be a subset of a space (X, τ). Then

(1) U is ω_δ-open if and only if it is ω-open and ω_δ^ω-open.
(2) U is δ-open if and only if it is ω_δ-open and ω_δ^δ-open.
(3) U is θ-open if and only if it is ω_δ-open and ω_δ^θ-open.
(4) U is θ-open if and only if it is ω_θ-open and ω_δ^θ-open.

Proof. (1) Let U be an ω_δ-open set. Then U is both ω-open and ω_δ^ω-open.
Conversely, let U be an ω-open and ω_δ^ω-open. Then $A = int_\omega A = \omega_\delta int A$, and hence by part (6) of Lemma 3.16, A is an ω_δ-open set.
The proof of the other parts are similar to the proof of part (1). □

THEOREM 3.22. Let U be a subset of a space (X, τ). Then

(1) If U is open and ω_δ^o-open, then it is ω_δ-open.
(2) If U is ω_δ-open and ω_δ^o-open, then it is open.
(3) U is open and ω_δ^o-open if and only if it is ω_δ-open and ω_δ^o-open.

Proof. Obvious. □

Corollary 3.23. *An ω_δ^o-open subset U of a space X is open if and only if it is ω_δ-open.*

The following results are characterizations of semi-regular and regular spaces.

Theorem 3.24. Let (X,τ) be any space. Then (X,τ) is a semi-regular space if and only if every open subset of X is both ω_δ^o-open and ω_δ^δ-open.

Proof. Let X be a semi-regular space, then $\tau_\delta = \tau$. Now, if G is open, then $intG = int_\delta G = \omega_\delta intG$, and hence it is both ω_δ^o-open and ω_δ^δ-open.
Conversely; let G be any open subset of X. Then $intG = G$, and by our hypothesis, the set G is both ω_δ^o-open and ω_δ^δ-open. Then $\omega_\delta intG = intG$ and $\omega_\delta intG = int_\delta G$. Thus $G = int_\delta G$. Hence $\tau_\delta = \tau$. Therefore, X is a semi-regular space. □

Theorem 3.25. Let (X,τ) be any space. Then (X,τ) is a regular space if and only if every open subset of X is both ω_δ^o-open and ω_δ^θ-open.

Proof. It is similar to the proof of the above result. □

4. ω_δ-Continuous functions and decompositions of some types of continuity

Definition 4.1. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be an ω_δ-continuous function if the inverse image of each open subset of Y is an ω_δ-open subset of X.

Theorem 4.2. For a function $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent:

(1) The inverse image of each closed subset of Y is an ω_δ-closed subset of X.
(2) $f(\omega_\delta ClA) \subseteq Clf(A)$ for each subset A of X.
(3) $\omega_\delta Clf^{-1}(B) \subseteq f^{-1}(ClB)$ for each subset B of Y.
(4) $f^{-1}(intB) \subseteq \omega_\delta intf^{-1}(B)$ for each subset B of Y.
(5) $f:(X,\omega_\delta O(X))\to(Y,\sigma)$ is continuous.
(6) For each $x\in X$ and each open subset G of Y containing $f(x)$, there exists an ω_δ-open subset U of X containing x such that $f(U)\subseteq G$.

Proof. Straightforward. □

Theorem 4.3. Let $f:(X,\tau) \to (Y,\sigma)$ be an onto ω_δ-continuous function. If $(X, \omega_\delta O(X))$ is Lindelöf, then Y is Lindelöf.

Proof. Let (X,τ) be a space for which $(X, \omega_\delta O(X))$ is a Lindelöf space. Let $\Psi = \{G_\lambda; \lambda \in \Lambda\ \}$ be any open cover of Y. Since f is ω_δ-continuous, then $f^{-1}(G_\lambda)$ is an ω_δ-open subset of X for each $\lambda \in \Lambda$. Thus $\{f^{-1}(G_\lambda);\ \lambda \in \Lambda\}$ is an ω_δ-open cover of X. Since $(X, \omega_\delta O(X))$ is a Lindelöf space, then there exits a countable subset Λ_0 of Λ such that $X = \bigcup\limits_{\lambda\in\Lambda_0} f^{-1}(G_\lambda)$. Since f is an onto function, then $Y = f(X) = \bigcup\limits_{\lambda\in\Lambda_0} f(f^{-1}(G_\lambda)) = \bigcup\limits_{\lambda\in\Lambda_0} (G_\lambda)$. Thus Ψ has a countable subcover. Hence Y is Lindelöf. $\square$

Corollary 4.4. *Let $f : X \to Y$ be an onto ω_δ-continuous function. If X is Lindelöf, then so is Y.*

We recall the following definitions:

Definition 4.5. A function $f : X \to Y$ is said to be a super-continuous [14](resp. clopen-continuous [12], Strongly θ-continuous (briefly, st.θ-continuous) [13, 10], ω-continuous [8], ω^o-continuous [2] and ω_θ-continuous [5]) function, if the inverse image of each open subset of Y is a δ-open (resp. clopen, θ-open, ω-open, ω^o-open and ω_θ-open) subset of X.

Remark 4.6. It is easy to see that:

(1) Every clopen-continuous, super-continuous, st.θ-continuous and ω_θ-continuous function is ω_δ-continuous.
(2) Every ω_δ-continuous function is ω-continuous and ω^o-continuous.
(3) Every ω_θ-continuous function is ω_δ-continuous.

The converse of neither part of the above remark is true. Also, the ω_δ-continuity and continuity are independent concepts, as the following examples show:

Example 4.7. Let (X,τ) be the space of Example 3.2 and let $\Im_{dis}$ be the discrete topology on X. Then the identity function $f : (X,\tau) \to (X, \Im_{dis})$ is an ω_δ-continuous, ω_θ-continuous, ω^o-continuous and ω- continuous but it is neither continuous, clopen-continuous, super-continuous nor st.θ-continuous.

Example 4.8. Let $(\mathbb{R},\tau)$ be the space of Example 3.3. Then the identity function $f : (\mathbb{R},\tau) \to (\mathbb{R},\tau)$ is continuous, ω^o-continuous and

ω-continuous but it is neither ω_δ-continuous, st.θ-continuous nor δ- continuous.

Example 4.9. Let $(\mathbb{R}, \tau)$ be the space of Example 3.4. Then the identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ is ω_δ-continuous but not ω_θ-continuous.

Definition 4.10. A function $f : X \to Y$ is said to be an ω_δ^ω-continuous (resp. ω_δ^δ-continuous, ω_δ^o-continuous and ω_δ^θ-continuous), if the inverse image of each open subset of Y is an ω_δ^ω-open (resp. ω_δ^δ-open, ω_δ^o-open and ω_δ^θ-open) subset of X.

Remark 4.11. It is easy to see that

(1) Every ω_δ-continuous function is ω_δ^ω-continuous.
(2) Every super-continuous function is ω_δ^δ-continuous, ω_δ^o-continuous and ω_δ^ω-continuous.
(3) Every st.θ-continuous function is ω_δ^θ-continuous, ω_δ^δ-continuous, ω_δ^o-continuous and ω_δ^ω-continuous.
(4) Every ω_δ^θ-continuous function is ω_δ^δ-continuous.

Example 4.12. (1) The identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \Im)$, where $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^c\}$ and $\Im = \{\phi, \mathbb{R}, \mathbb{N}\}$ is (ω_δ^δ, ω_δ^ω, ω_δ^o and ω_δ^θ)-continuous but it is neither continuous nor (copen, super, st.θ ω_δ, ω^o nor ω)-continuous.

(2) The identity function $f : (X, \tau) \to (X, \tau)$, where (X, τ) is the space of Example 3.2 is continuous and (ω_δ^o, ω_δ^ω and ω_δ)-continuous but it is neither (ω_δ^δ nor ω_δ^θ)-continuous.
(3) The identity function $f : (\mathbb{R}, \tau_{coc}) \to (\mathbb{R}, \tau_{coc})$ is continuous and ω_δ^δ-continuous but it is neither (super, st.θ, ω_δ, ω_δ^o nor ω_δ^θ)-continuous.
(4) The identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$, where $(\mathbb{R}, \tau)$ is the space of Example 3.19 is ω-continuous but not ω_δ-continuous.
(5) The identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$, where $(\mathbb{R}, \tau)$ is the space that given in Example 3.4 is (ω_δ and ω_δ^δ)-continuous but it is not ω_δ^θ-continuous.

The next result is the following decompositions of some types of continuity

Theorem 4.13. Let $f : X \to Y$ be a function. Then

(1) f is ω_δ-continuous if and only if it is ω-continuous and ω_δ^ω-continuous.
(2) f is super-continuous if and only if it is ω_δ-continuous and ω_δ^δ-continuous.

(3) f is st.θ-continuous if and only if it is ω_δ-continuous and ω_δ^θ-continuous.
(4) f is st.θ-continuous if and only if it is ω_θ-continuous and ω_δ^θ-continuous.

Proof. (1) Let f be an ω_δ-continuous function. Let G be any open subset of Y. Then $f^{-1}(G)$ is an ω_δ-open subset of X. So by part (1) of Theorem 3.21 G is both ω-open and ω_δ^ω-open. Thus f is both ω-continuous and ω_δ^ω-continuous. Conversely; let f be a function which is both ω-continuous and ω_δ^ω-continuous. If G is any open subset of Y, then $f^{-1}(G)$ is both ω-open and ω_δ^ω-open. So by part (1) of Theorem 3.21 G is an ω_δ-open subset of X. Hence f is ω_δ-continuous.
The proof of other parts are similar to the proof of part (1). □

Theorem 4.14. Let $f : X \to Y$ be a function. Then
(1) If f is continuous and ω_δ^o-continuous, then f is ω_δ-continuous.
(2) If f is ω_δ-continuous and ω_δ^o-continuous, then it is continuous.
(3) f is continuous and ω_δ^o-continuous if and only if ω_δ-continuous and ω_δ^o-continuous.

Proof. It follows from Theorem 3.22. □

Definition 4.15. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω_δ^*-continuous if $f^{-1}(Fr(G))$ is ω_δ-closed for each open subset G of Y, where $Fr(G) = ClG - G$.

It is easy to see that each ω_δ-continuous function is ω_δ^*-continuous, but not conversely as the following example shows:

Example 4.16. Let $f : (\mathbb{R}, \tau_{coc}) \to (Y, \Im_{dis})$, where $Y = \{a, b\}$ be a function given by $f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q} \\ b, & \text{if } x \in \mathbb{Q}^c \end{cases}$. Then f is ω_δ^*-continuous but it is not ω_δ-continuous.

Definition 4.17. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly ω_δ-continuous (simply, wω_δ-continuous) if $f^{-1}(G) \subseteq \omega_\delta int f^{-1}(ClG)$.

It is easy to see that each ω_δ-continuous function is wω_δ-continuous, but not conversely as the following example shows:

Example 4.18. Let $Y = \{a, b, c\}$ and $\Im = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\}$ and let $f : (\mathbb{R}, \tau_{coc}) \to (Y, \Im)$ be a function given by $f(x) = \begin{cases} a, & \text{if } x \in Q \\ c, & \text{if } x \in \mathbb{Q}^c \end{cases}$.
Then f is wω_δ-continuous but it is neither ω_δ-continuous nor ω_δ^*-continuous.

Since the function which given in Example 4.16, is ω_δ^*-continuous but not wω_δ-continuous. Therefore, ω_δ^*-continuity and wω_δ-continuity are independent concepts. Thus we obtain the following diagram, where by "cont." we mean "continuous"

$$\begin{array}{ccccccc}
 & & cont. & \rightarrow & \omega^o - cont. & \rightarrow & \omega - cont. \\
\omega_\delta^o - cont. & & \uparrow & & \uparrow & & w\omega_\delta - cont. \\
 & \nwarrow & \uparrow & & \uparrow & \nearrow & \\
\omega_\delta^\delta - cont. & \leftarrow & super - cont. & \rightarrow & \omega_\delta - cont. & \rightarrow & \omega_\delta^* - cont. \\
\uparrow & & \uparrow & & \uparrow & \searrow & \\
\omega_\delta^\theta - cont. & \leftarrow & st.\theta - cont. & \rightarrow & \omega_\theta - cont. & & \omega_\delta^\omega - cont. \\
 & & \uparrow & & & & \\
 & & clopen - cont. & & & &
\end{array}$$

Our final result is the following decompositions of ω_δ-continuity:

THEOREM 4.19. A function $f : X \to Y$ is ω_δ-continuous if and only if it is wω_δ-continuous and ω_δ^*-continuous.

Proof. The part ω_δ-continuity implies wω_δ-continuity and ω_δ^*-continuity is obvious. Conversely, suppose that f is both wω_δ-continuous and ω_δ^*-continuous. To show f is ω_δ-continuous. Let G be any open subset of Y. Then by wω_δ-continuity of f, we have $f^{-1}(G) \subseteq \omega_\delta int f^{-1}(ClG)$ and by ω_δ^*-continuity of f, we have $f^{-1}(Fr(G))$ is an ω_δ-closed subset of X. Since $f^{-1}(G) \cap f^{-1}(Fr(G) = \phi$, then $f^{-1}(G) \subseteq X - f^{-1}(Fr(G))$. Since $X - f^{-1}(Fr(G))$ is ω_δ-open, then by Lemma 3.16, $f^{-1}(G) \subseteq \omega_\delta int(X - f^{-1}(Fr(G)))$ and since $G = ClG - Fr(G)$, then $f^{-1}(G) \subseteq \omega_\delta int f^{-1}(ClG) \cap \omega_\delta int(X - f^{-1}(Fr(G))) = \omega_\delta int f^{-1}(ClG - Fr(G)) = \omega_\delta int f^{-1}(G)$. Hence by Lemma 3.16 $f^{-1}(G)$ is an ω_δ-open subset of X. Thus f is ω_δ-continuous. □

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Department of Mathematics
School of Science
Faculty of Science and Education Science
University of Sulaimani
Sulaimani, Kurdistan Region, Iraq
E-mail: darweshymath@yahoo.com